THE COLLEGES OF OXFORD UNIVERSITY
MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE

WEDNESDAY 31 OCTOBER 2007

Time allowed: $2\frac{1}{2}$ hours

For candidates applying for Mathematics, Mathematics & Statistics, Computer Science, Mathematics & Computer Science, or Mathematics & Philosophy

---

Write your name, test centre (where you are sitting the test), Oxford college (to which you have applied or been assigned) and your proposed course (from the list above) in BLOCK CAPITALS

NAME: MODEL ANSWERS
TEST CENTRE:
OXFORD COLLEGE (if known):
DEGREE COURSE:
DATE OF BIRTH: **Special Arrangements:** [ ]

NOTE: This paper contains 7 questions, of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

Mathematics / Maths & Philosophy / Maths & Statistics candidates should attempt **Questions 1, 2, 3, 4, 5.**

Maths & Computer Science candidates should attempt **Questions 1, 2, 3, 5, 6.**

Computer Science candidates should attempt **Questions 1, 2, 5, 6, 7.**

Further credit cannot be gained by attempting extra questions.

Question 1 is a multiple choice question with ten parts, for which marks are given solely for the correct answers, though you may use the space between parts for rough work. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to Questions 2–7 should be written in the space provided, continuing onto the blank pages at the end of this booklet if necessary. Each of Questions 2–7 is worth 15 marks.

ONLY ANSWERS WRITTEN IN THIS BOOKLET WILL BE MARKED.
DO NOT INCLUDE EXTRA SHEETS OR ROUGH WORK.

THE USE OF CALCULATORS, FORMULA SHEETS
AND DICTIONARIES IS PROHIBITED.

1. For **ALL APPLICANTS**.

For each part of the question on pages 3–7 you will be given four possible answers, just one of which is correct. Indicate for each part **A**–**J** which answer (a), (b), (c), or (d) you think is correct with a tick (✓) in the corresponding column in the table below. Please show any rough working in the space provided between the parts.

	(a)	(b)	(c)	(d)
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

A. Let r and s be integers. Then

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}}$$

is an integer if

(a) $r + s \leqslant 0$,
(b) $s \leqslant 0$,
(c) $r \leqslant 0$,
(d) $r \geqslant s$.

$$\frac{6^{r+s} \times 12^{r-s}}{8^r \times 9^{r+2s}} = \frac{2^{(r+s+2r-2s)} \times 3^{(r+s+r-s)}}{2^{3r} \times 3^{2r+4s}}$$

$$= 2^{-s} \times 3^{-4s}$$

if $s \leqslant 0$, $2^{-s} \times 3^{-4s}$ will be an integer

B. The greatest value which the function

$$f(x) = \left(3\sin^2(10x + 11) - 7\right)^2$$

takes, as x varies over all real values, equals

(a) -9, (b) 16, (c) 49, (d) 100.

$-1 \leqslant \sin(10x + 11) \leqslant 1$

$0 \leqslant \sin^2(10x + 11) \leqslant 1$

$0 \leqslant 3\sin^2(10x + 11) \leqslant 3$

$-7 \leqslant (3\sin^2(10x + 11) - 7) \leqslant -4$

$16 \leqslant (3\sin^2(10x + 11) - 7)^2 \leqslant 49$

max value = 49

C. The number of solutions x to the equation

$$7\sin x + 2\cos^2 x = 5,$$

in the range $0 \leqslant x < 2\pi$, is

(a) 1, (b) 2, (c) 3, (d) 4.

$7\sin x + 2\cos^2 x = 5$

$\cos^2 x + \sin^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

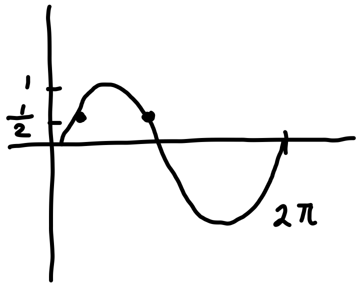
$7\sin x + 2 - 2\sin^2 x = 5$

$2\sin^2 x - 7\sin x + 3 = 0$

$(2\sin x - 1)(\sin x - 3) = 0$

$\sin x \neq 3$ no solutions

$\sin x = \frac{1}{2}$ 2 solutions



D. The point on the circle

$$(x-5)^2 + (y-4)^2 = 4$$

which is closest to the circle

$$(x-1)^2 + (y-1)^2 = 1$$

is

(a) $(3.4, 2.8)$, (b) $(3, 4)$, (c) $(5, 2)$, (d) $(3.8, 2.4)$.

line from centre $(5,4)$ to $(1,1)$ goes through closest point on each circle to the other

gradient $= \frac{4-1}{5-1} = \frac{3}{4}$ $\quad y - 1 = \frac{3}{4}(x-1)$ $\quad y = \frac{3}{4}x + \frac{1}{4}$

substitute in $x = 3.4$, $y = 2.8$ $\therefore$ a is on the line

$x = 3$, $y = 2.5 \neq 4$ $\therefore$ b is not on the line

$x = 5$, $y = 4 \neq 2$ $\therefore$ c is not on the line

$x = 3.8$, $y = 3.1 \neq 2.4$ $\therefore$ d is not on the line

E. If x and n are integers then

$$(1-x)^n (2-x)^{2n} (3-x)^{3n} (4-x)^{4n} (5-x)^{5n}$$

is

(a) negative when $n > 5$ and $x < 5$,
(b) negative when n is odd and $x > 5$,
(c) negative when n is a multiple of 3 and $x > 5$,
(d) negative when n is even and $x < 5$.

If n is odd and $x > 5$, the expression would be :

$(\text{negative})^{\text{odd}} \times (\text{negative})^{\text{even}} \times (\text{negative})^{\text{odd}} \times (\text{negative})^{\text{even}} \times (\text{negative})^{\text{odd}}$

$= \text{negative} \times \text{positive} \times \text{negative} \times \text{positive} \times \text{negative}$

$= \text{negative}$

F. The equation

$$8^x + 4 = 4^x + 2^{x+2}$$

has

(a) no real solutions;
(b) one real solution;
(c) two real solutions;
(d) three real solutions.

$8^x + 4 = 4^x + 2^{x+2}$

$2^{3x} + 2^2 = 2^{2x} + 2^2 \times 2^x$

$2^{2x}(2^x - 1) = 2^2(2^x - 1)$

$2^{2x} = 2^2 \qquad 2^x - 1 = 0$

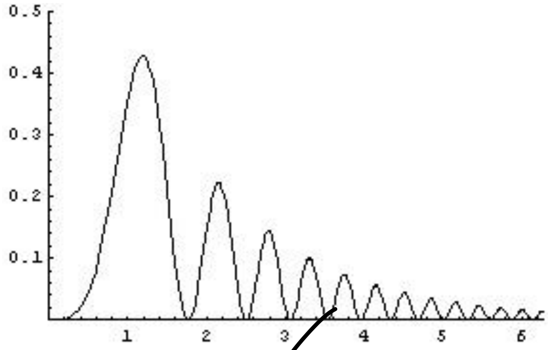
$2x = 2 \qquad 2^x = 1$

$x = 1 \qquad x = 0$

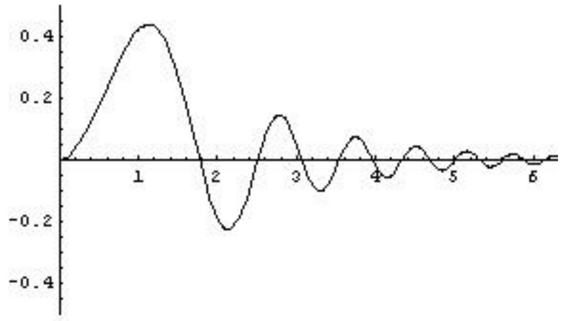
$\therefore$ two real solutions

G. On which of the axes below is a sketch of the graph

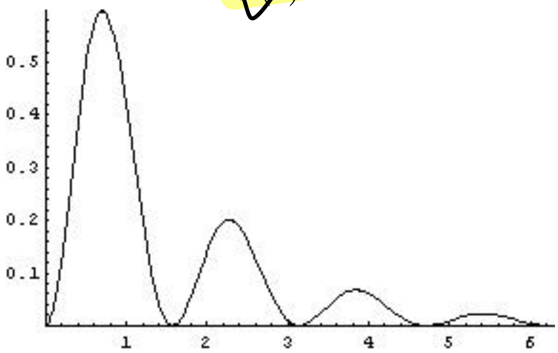
$$y = 2^{-x} \sin^2\left(x^2\right)?$$



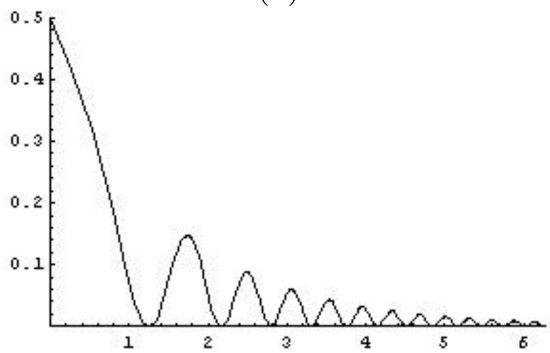
(a) ✓



(b)



(c)



(d)

$y = \frac{1}{2^x} \sin^2(x^2)$

Both $\frac{1}{2^x}$ and $\sin^2(x^2)$ are always positive, making y always positive and eliminating (b).

$x = 0$, $y = \frac{1}{2^0} \sin^2(0) = 0$, eliminating (d)

$y = 0$ when $x^2 = 0, \pi, 2\pi \ldots$

$(\sin^2(n\pi) = 0)$

$x = 0, \sqrt{\pi}, \sqrt{2\pi}$

eliminating (c)

$\therefore$ answer is (a)

H. Given a function $f(x)$, you are told that

$$\int_0^1 3f(x)\,\mathrm{d}x + \int_1^2 2f(x)\,\mathrm{d}x = 7,$$
$$\int_0^2 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x = 1.$$

It follows that $\int_0^2 f(x)\,\mathrm{d}x$ equals

(a) -1, (b) 0, (c) $\frac{1}{2}$, (d) 2.

$$3\int_0^1 f(x)\,dx + 2\int_1^2 f(x)\,dx = 7 \quad \ldots (1)$$

$$\int_0^2 f(x)\,dx + \int_1^2 f(x)\,dx = 1 \quad \ldots (2)$$

$(1) + (2)$: $\left(\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \int_0^2 f(x)\,dx\right)$

$$3\int_0^1 f(x)\,dx + 3\int_1^2 f(x)\,dx + \int_0^2 f(x)\,dx = 8$$

$$\Rightarrow 4\int_0^2 f(x)\,dx = 8$$

$$\Rightarrow \int_0^2 f(x)\,dx = 2$$

$$3\int_0^1 f(x)\,dx + 3\int_1^2 f(x)\,dx$$
$$= 3\left[\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx\right]$$
$$= 3\int_0^2 f(x)\,dx$$

I. Given that a and b are positive and

$$4\left(\log_{10} a\right)^2 + \left(\log_{10} b\right)^2 = 1,$$

then the greatest possible value of a is

(a) $\frac{1}{10}$, (b) 1, (c) $\sqrt{10}$, (d) $10^{\sqrt{2}}$.

$4(\log_{10} a)^2 + (\log_{10} b)^2 = 1$

minimum value of $(\log_{10} b)^2 = 0$, when $b = 1$

maximum value of a: $4(\log_{10} a)^2 = 1$

$(\log_{10} a)^2 = \frac{1}{4}$

$\log_{10} a = \frac{1}{2}$

$a = \sqrt{10}$

$(\log_{10} a)^2$ is at a maximum value when a is at a maximum value

J. The inequality

$$(n+1) + \left(n^4 + 2\right) + \left(n^9 + 3\right) + \left(n^{16} + 4\right) + \cdots + \left(n^{10000} + 100\right) > k$$

is true for all $n \geqslant 1$. It follows that

(a) $k < 1300$,
(b) $k^2 < 101$,
(c) $k \geqslant 101^{10000}$,
(d) $k < 5150$.

The equation can be rewritten as:

$n + n^4 + n^9 + \ldots + n^{10000} + 1 + 2 + 3 + \ldots + 100 > k$

$1 + 2 + 3 + \ldots + 100 = \frac{100}{2}(1 + 100) = 5050$

$n + n^4 + n^9 + \ldots + n^{10000} + 5050 > k$

minimum value of $n + n^4 \ldots + n^{10000}$ is when $n = 1$

$1 \times 100 + 5050 > k$

$k < 5150$

2. For **ALL APPLICANTS.**

Let

$$f_n(x) = \left(2 + (-2)^n\right) x^2 + (n+3)\,x + n^2$$

where n is a positive integer and x is any real number.

(i) Write down $f_3(x)$.

Find the maximum value of $f_3(x)$.

For what values of n does $f_n(x)$ have a maximum value (as x varies)?

[Note you are not being asked to calculate the value of this maximum.]

(ii) Write down $f_1(x)$.

Calculate $f_1(f_1(x))$ and $f_1(f_1(f_1(x)))$.

Find an expression, simplified as much as possible, for

$$f_1(f_1(f_1(\cdots f_1(x))))$$

where f_1 is applied k times. [Here k is a positive integer.]

(iii) Write down $f_2(x)$.

The function

$$f_2(f_2(f_2(\cdots f_2(x)))),$$

where f_2 is applied k times, is a polynomial in x. What is the degree of this polynomial?

2 (i) $f_n(x) = (2+(-2)^n)x^2 + (n+3)x + n^2$

For $n=3$: $f_3(x) = (2+(-2)^3)x^2 + (3+3)x + (3)^2$

$= (2+-8)x^2 + 6x + 9$

$= -6x^2 + 6x + 9$

$= -6\left[x^2 - x - \frac{3}{2}\right]$

$= -6\left[(x-\frac{1}{2})^2 - \frac{1}{4} - \frac{3}{2}\right] = -6\left[(x-\frac{1}{2})^2 - \frac{7}{4}\right]$

$= -6(x-\frac{1}{2})^2 + \frac{21}{2}$

So, $f_3(x) = -6(x-\frac{1}{2})^2 + \frac{21}{2}$. //

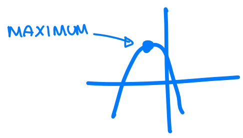
The maximum of $f_3(x)$ is $\frac{21}{2}$ which occurs at $x = \frac{1}{2}$. //

In general, $f_n(x)$ is a quadratic in x which has a maximum when the leading coefficient is negative.

$2 + (-2)^n < 0 \Rightarrow (-2)^n < -2$. This inequality is only true for odd numbers greater than 1.

For even numbers LHS would be positive which cannot then be less than -2. For $n=1$ we have equality.

This coincides with the quadratic graph having shape $\cap$:



2(ii) $f_n(x) = (2+(-2)^n)x^2 + (n+3)x + n^2$

For $n=1$: $f_1(x) = (2+(-2)^1)x^2 + (1+3)x + 1^2$
$= 4x+1$

So, $f_1(x) = 4x+1$. //

$$f_1(f_1(x)) = 4f_1(x)+1 = 4(4x+1)+1 = 16x+4+1 = 16x+5$$

$$f_1(f_1(f_1(x))) = 4f_1(f_1(x))+1 = 4(16x+5)+1 = 64x+20+1 = 64x+21$$

(using $f_1(f_1(x))$ from above)

$$f_1^k(x) = \underbrace{f_1(f_1(f_1(\cdots f_1(x))))}_{k \text{ times}}$$

$$= 4f_1^{k-1}(x)+1$$

$$= 4\cdot\left[4(f_1^{k-2}(x))+1\right]+1 = 4^2 f_1^{k-2}(x) + 4 + 1$$

Notice these powers sum to k

$$= 4^2\left[4(f_1^{k-3}(x))+1\right]+4+1$$

$$= 4^3 f_1^{k-3}(x) + 4^2 + 4 + 1$$

$$\vdots$$

$$= 4^{k-1}f_1(x) + 4^{k-2} + 4^{k-3} + \dots + 4^2 + 4 + 1$$

$$= 4^{k-1}(4x+1) + 4^{k-2} + 4^{k-3} + \dots + 4^2 + 4 + 1$$

$$= 4^k x + 4^{k-1} + 4^{k-2} + 4^{k-3} + \dots + 4^2 + 4 + 1$$

$$= 4^k x + (1 + 4 + 4^2 + \dots + 4^{k-1})$$

$$= 4^k x + \left(\frac{1(1-4^k)}{1-4}\right) = 4^k x + \frac{4^k - 1}{3} \quad //$$

using sum of finite geometric series: $S_n = \frac{a_1(1-r^n)}{1-r}$
where,
$a_1 = 1$
$r = 4$
$n = k$ (the number of terms)

2(iii). $f_n(x) = (2+(-2)^n)x^2 + (n+3)x + n^2$

For $n=2$: $f_2(x) = (2+(-2)^2)x^2 + (2+3)x + 2^2$
$\Rightarrow f_2(x) = 6x^2+5x+4$ // This is a polynomial of degree 2.

For $k=2$: $f_2(f_2(x)) = 6(6x^2+5x+4)^2 + 5(6x^2+5x+4)+4$
We can see the degree will be 4 (which comes from squaring the x^2 term).
So, in general, the degree will double at each iteration, i.e. $f_2(f_2(f_2(x)))$ will be degree $2^3 = 8$.
So in general, $f_2(f_2(f_2(\dots f_2(x))))$ k times will be degree 2^k.

3.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{MATHEMATICS} \\ \textbf{MATHEMATICS \& STATISTICS} \\ \textbf{MATHEMATICS \& PHILOSOPHY} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE} \end{array} \right\}$ **ONLY.**

Computer Science **applicants should turn to page 14.**

Let

$$I(c) = \int_0^1 \left((x-c)^2 + c^2\right) \mathrm{d}x$$

where c is a real number.

(i) Sketch $y = (x-1)^2 + 1$ for the values $-1 \leqslant x \leqslant 3$ on the axes below and show on your graph the area represented by the integral $I(1)$.

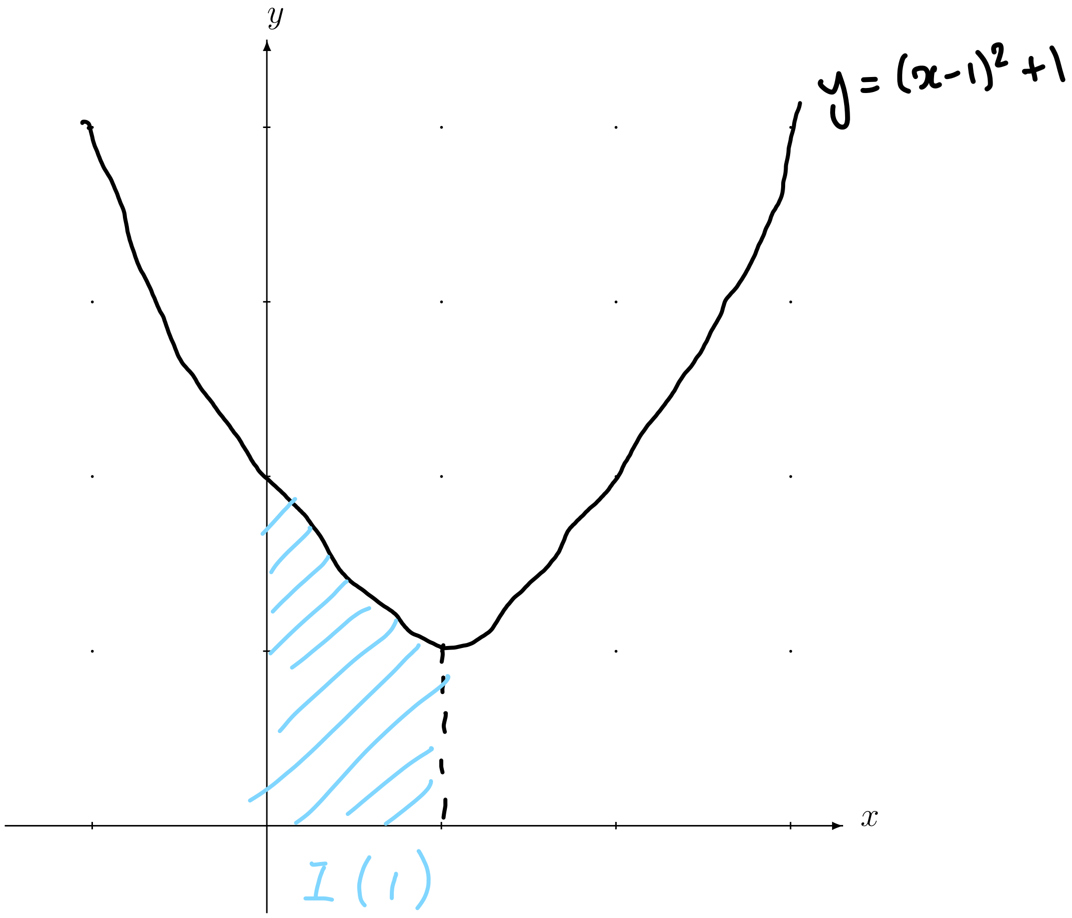
(ii) Without explicitly calculating $I(c)$, explain why $I(c) \geqslant 0$ for any value of c.

(iii) Calculate $I(c)$.

(iv) What is the minimum value of $I(c)$ (as c varies)?

(v) What is the maximum value of $I(\sin\theta)$ as θ varies?

3 i)



3ii) both $(x-c)^2$ and c^2 are positive, meaning $((x-c)^2+c^2) \geq 0$ and $\therefore I(c) \geq 0$

3iii) $\int_0^1 x^2 - 2cx + 2c^2 \, dx = \left[\frac{x^3}{3} - cx^2 + 2c^2x\right]_0^1$

$$I(c) = \frac{1}{3} - c + 2c^2 - 0$$

$$I(c) = 2c^2 - c + \frac{1}{3}$$

iv) $I'(c) = 4c - 1 = 0$ $\quad (I''(c) = 4 > 0$, minimum point$)$

$c = \frac{1}{4}$

$I\left(\frac{1}{4}\right) = 2 \times \left(\frac{1}{4}\right)^2 - \frac{1}{4} + \frac{1}{3}$

$= \frac{5}{24}$ = minimum value of $I(c)$

v) $-1 \leq \sin\theta \leq 1$ maximum value is at $I(-1)$, the furthest away from the minimum point $(c = \frac{1}{4})$ is $c = -1$

$I(-1) = 2 \times 1 + 1 + \frac{1}{3}$

$= 3 + \frac{1}{3}$

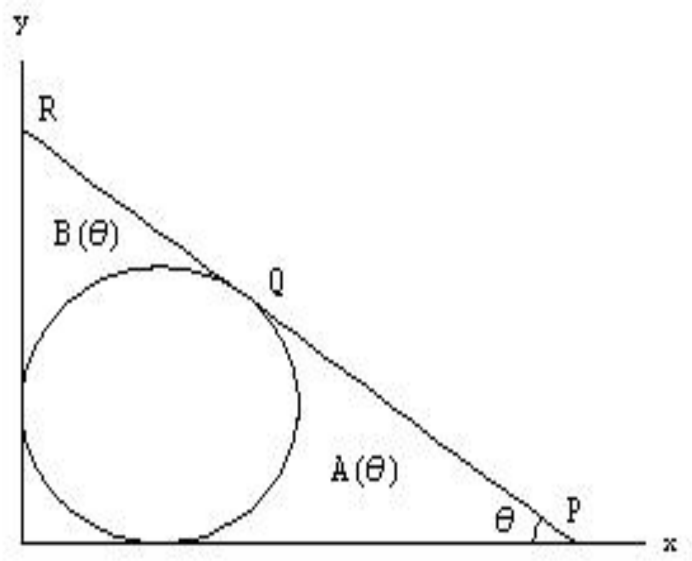
$= \frac{10}{3}$

4.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{MATHEMATICS} \\ \textbf{MATHEMATICS \& STATISTICS} \\ \textbf{MATHEMATICS \& PHILOSOPHY} \end{array} \right\}$ **ONLY.**

Mathematics & Computer Science **and** *Computer Science* **applicants should turn to page 14.**

In the diagram below is sketched the circle with centre $(1, 1)$ and radius 1 and a line L. The line L is tangential to the circle at Q; further L meets the y-axis at R and the x-axis at P in such a way that the angle OPQ equals θ where $0 < \theta < \pi/2$.



(i) Show that the co-ordinates of Q are

$$(1 + \sin\theta, 1 + \cos\theta)\,,$$

and that the gradient of PQR is $-\tan\theta$.

Write down the equation of the line PQR and so find the co-ordinates of P.

(ii) The region bounded by the circle, the x-axis and PQ has area $A(\theta)$; the region bounded by the circle, the y-axis and QR has area $B(\theta)$. (See diagram.)

Explain why

$$A(\theta) = B(\pi/2 - \theta)$$

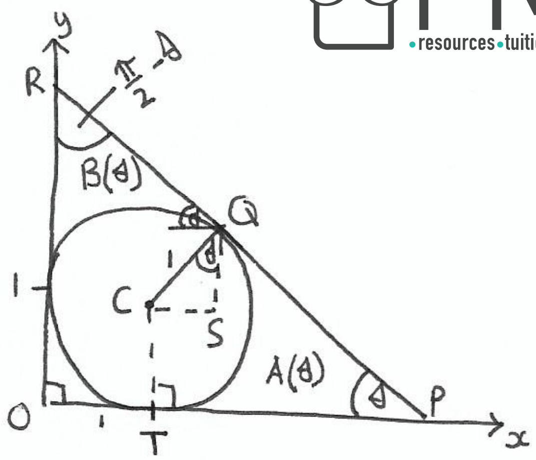
for any θ.

Calculate $A(\pi/4)$.

(iii) Show that

$$A\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{\pi}{3}.$$

4



i. $Q(x, y)$

$\sin\theta = \frac{CS}{1}$ $\cos\theta = \frac{SQ}{1}$

$x = 1 + CS$ $y = 1 + SQ$

$x = 1 + \sin\theta$ $y = 1 + \cos\theta$

$Q(1+\sin\theta,\ 1+\cos\theta)$

CQ is perpendicular to PQR

gradient of $CQ = \frac{SQ}{CS} = \frac{\cos\theta}{\sin\theta}$

$-1 \div \frac{\cos\theta}{\sin\theta}$ = gradient of $PQR = -\tan\theta$

$$y - (1+\cos\theta) = -\tan\theta(x - (1+\sin\theta))$$

at P, $y = 0$

$$-\frac{(1+\cos\theta)}{-\tan\theta} = x - (1+\sin\theta)$$

$$\frac{\cos\theta + \cos^2\theta}{\sin\theta} + \frac{\sin\theta + \sin^2\theta}{\sin\theta} = x$$

$$x = \frac{1+\cos\theta+\sin\theta}{\sin\theta} \quad P\left(\frac{1+\cos\theta+\sin\theta}{\sin\theta},\ 0\right)$$

ii. $\angle ORP = \frac{\pi}{2} - \theta$

$\therefore$ the area $B(\theta)$ would be written as $A\left(\frac{\pi}{2}-\theta\right)$ and the area $A(\theta)$ would be written as $B\left(\frac{\pi}{2}-\theta\right)$

$\therefore A(\theta) = B\left(\frac{\pi}{2}-\theta\right)$

4ii. (Continued) $\theta = \frac{\pi}{4}$

$$A\left(\frac{\pi}{4}\right) = B\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = B\left(\frac{\pi}{4}\right) \quad \therefore \text{at } \theta = \frac{\pi}{4},\ OP = OR$$

$$A\left(\frac{\pi}{4}\right) + B\left(\frac{\pi}{4}\right) + \frac{3\pi}{4} \times 1^2 + 1 \times 1 = \frac{1}{2} \times OP \times OR$$

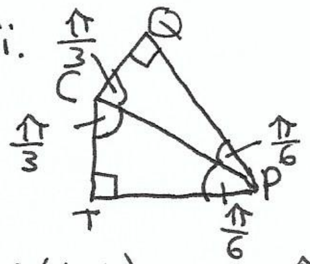
$$2A\left(\frac{\pi}{4}\right) + \frac{3\pi}{4} + 1 = \frac{1}{2} \times \left(\frac{1 + \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)}\right)^2$$

$$2A\left(\frac{\pi}{4}\right) + \frac{3\pi}{4} + 1 = \frac{1}{2} \times \left(2 + \sqrt{2}\right)^2 = \frac{1}{2}\left(4 + 2 + 4\sqrt{2}\right)$$

$$2A\left(\frac{\pi}{4}\right) = 2 + 2\sqrt{2} - \frac{3\pi}{4}$$

$$A\left(\frac{\pi}{4}\right) = 1 + \sqrt{2} - \frac{3\pi}{8}$$

iii.



$$\text{Area of } OPQ = \frac{2\pi}{3} \times \frac{1}{2\pi} \times \pi + \frac{1}{2} + A\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} + \frac{1}{2} + A\left(\frac{\pi}{3}\right) \quad (1)$$

$$\text{Area of } OPQ = (OP - 1) \times 1 \times \frac{1}{2} \times 2 + \frac{1}{2}$$

$$= \sqrt{3} + \frac{1}{2} \quad (2)$$

$$OP\left(\text{at } \frac{\pi}{3}\right) = \frac{1 + \cos\frac{\pi}{3} + \sin\frac{\pi}{3}}{\sin\frac{\pi}{3}}$$

$$= 1 + \sqrt{3}$$

$(1) = (2)$

$$\frac{\pi}{3} + \frac{1}{2} + A\left(\frac{\pi}{3}\right) = \sqrt{3} + \frac{1}{2}$$

$$A\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{\pi}{3}$$

5. For **ALL APPLICANTS.**

Let $f(n)$ be a function defined, for any integer $n \geqslant 0$, as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ (f(n/2))^2 & \text{if } n > 0 \text{ and } n \text{ is even}, \\ 2f(n-1) & \text{if } n > 0 \text{ and } n \text{ is odd}. \end{cases}$$

(i) What is the value of $f(5)$?

The *recursion depth* of $f(n)$ is defined to be the number of other integers m such that the value of $f(m)$ is calculated whilst computing the value of $f(n)$. For example, the recursion depth of $f(4)$ is 3, because the values of $f(2), f(1)$, and $f(0)$ need to be calculated on the way to computing the value of $f(4)$.

(ii) What is the recursion depth of $f(5)$?

Now let $g(n)$ be a function, defined for all integers $n \geqslant 0$, as follows:

$$g(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 + g(n/2) & \text{if } n > 0 \text{ and } n \text{ is even}, \\ 1 + g(n-1) & \text{if } n > 0 \text{ and } n \text{ is odd}. \end{cases}$$

(iii) What is $g(5)$?

(iv) What is $g(2^k)$, where $k \geqslant 0$ is an integer? Briefly explain your answer.

(v) What is $g(2^l + 2^k)$ where $l > k \geqslant 0$ are integers? Briefly explain your answer.

(vi) Explain briefly why the value of $g(n)$ is equal to the recursion depth of $f(n)$.

i) $f(5) = 2f(4)$

$f(4) = (f(2))^2$

$f(2) = (f(1))^2$

$f(1) = 2f(0)$

$f(0) = 1$

$f(1) = 2$

$f(2) = 4$

$f(4) = 16$

$f(5) = 32$

ii) 4 (the values of $f(4), f(2), f(1)$ and $f(0)$ are required to calculate $f(5)$)

iii) $g(5) = 1 + g(4)$

$g(4) = 1 + g(2)$

$g(2) = 1 + g(1)$

$g(1) = 1 + g(0)$

$g(0) = 0$

$g(1) = 1$

$g(2) = 2$

$g(4) = 3$

$g(5) = 4$

iv) $g(2^k) = 1 + g(2^{k-1}) = k \times 1 + g(2^0)$

$g(2^k) = k + g(1)$

$g(2^k) = k + 1$

v) $g(2^l + 2^k) = k + g(2^0 + 2^{l-k})$

$g(2^l + 2^k) = k + 1 + g(2^{l-k})$

$g(2^l + 2^k) = k + 1 + l - k + 1$

$g(2^l + 2^k) = l + 2$

vi) for $g(n)$, no matter whether n is odd or even, 1 is added to previous value. As $g(0) = 0$, $g(n) =$ number of already calculated values

$\therefore g(n) =$ recursion depth

6.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{COMPUTER SCIENCE} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE} \end{array} \right\}$ **ONLY.**

Three people called Alf, Beth, and Gemma, sit together in the same room.

One of them always tells the truth.
One of them always tells a lie.
The other one tells truth or lies at random.

In each of the following situations, your task is determine how each person acts.

(i) Suppose that Alf says "I always tell lies" and Beth says "Yes, that's true, Alf always tells lies".

Who always tells the truth? Who always lies? Briefly explain your answer.

(ii) Suppose instead that Gemma says "Beth always tells the truth" and Beth says "That's wrong."

Who always tells the truth? Who always lies? Briefly explain your answer.

(iii) Suppose instead that Alf says "Beth is the one who behaves randomly" and Gemma says "Alf always lies". Then Beth says "You have heard enough to determine who always tells the truth".

Who always tells the truth? Who always lies? Briefly explain your answer.

6.i. The statement "I always tell lies" cannot be said by the one always telling the truth (then the statement would be a lie) or the one always lying (then the statement would be true). ∴ Alf tells truth or lies at random. Beth lies when she says Alf always tells lies, meaning she always lies and Gemma always tells the truth.

ii. If Beth always tells the truth, her contradiction would be a lie. ∴ Gemma's statement is a lie, meaning Alf must be the one who always tells the truth. Beth's statement that she does not always tell the truth is true, making her the random person and Gemma the one who always lies.

iii. Considering the first two statements, if Alf always tells the truth, Beth is the random person and Gemma always lies. If Alf is the random person, Gemma always lies and Beth always tells the truth. If Alf always lies, Beth always tells the truth and Gemma is the random person. Therefore, based only on the first two statements, there are three possibilities, all of which could work. Therefore, we have not heard enough to determine who always tells the truth (it could be Alf or Beth) so Beth's statement is a lie. This means she cannot always tell the truth, so Alf always tells the truth, Beth is the random person and Gemma always lies.

7. For **APPLICANTS IN COMPUTER SCIENCE ONLY.**

Suppose we have a collection of tiles, each containing two strings of letters, one above the other. A **match** is a list of tiles from the given collection (tiles may be used repeatedly) such that the string of letters along the top is the same as the string of letters along the bottom. For example, given the collection

$$\left\{ \left[\frac{\mathsf{AA}}{\mathsf{A}}\right], \left[\frac{\mathsf{B}}{\mathsf{ABA}}\right], \left[\frac{\mathsf{CCA}}{\mathsf{CA}}\right] \right\},$$

the list

$$\left[\frac{\mathsf{AA}}{\mathsf{A}}\right] \left[\frac{\mathsf{B}}{\mathsf{ABA}}\right] \left[\frac{\mathsf{AA}}{\mathsf{A}}\right]$$

is a match since the string AABAA occurs both on the top and bottom.

Consider the following set of tiles:

$$\left\{ \left[\frac{\mathsf{X}}{\mathsf{U}}\right], \left[\frac{\mathsf{UU}}{\mathsf{U}}\right], \left[\frac{\mathsf{Z}}{\mathsf{X}}\right], \left[\frac{\mathsf{E}}{\mathsf{ZE}}\right], \left[\frac{\mathsf{Y}}{\mathsf{U}}\right], \left[\frac{\mathsf{Z}}{\mathsf{Y}}\right] \right\}.$$

(a) What tile must a match begin with?

(b) Write down all the matches which use four tiles (counting any repetitions).

(c) Suppose we replace the tile $\left[\frac{\mathsf{E}}{\mathsf{ZE}}\right]$ with $\left[\frac{\mathsf{E}}{\mathsf{ZZZE}}\right]$.

What is the least number of tiles that can be used in a match?

How many different matches are there using this smallest numbers of tiles?

[Hint: you may find it easiest to construct your matches backwards from right to left.]

Consider a new set of tiles $\left\{ \left[\frac{\mathsf{XXXXXXX}}{\mathsf{X}}\right], \left[\frac{\mathsf{X}}{\mathsf{XXXXXXXXXX}}\right] \right\}$. (The first tile has seven Xs on top, and the second tile has ten Xs on the bottom.)

(d) For which numbers n do there exist matches using n tiles? Briefly justify your answer.

7a. $\left[\frac{UU}{U}\right]$

b. $\left[\frac{UU}{U}\right]\left[\frac{X}{U}\right]\left[\frac{Z}{X}\right]\left[\frac{E}{ZE}\right]$

$\left[\frac{UU}{U}\right]\left[\frac{Y}{U}\right]\left[\frac{Z}{Y}\right]\left[\frac{E}{ZE}\right]$

c. 10 tiles

$\left[\frac{UU}{U}\right]\left[\frac{UU}{U}\right]\left[\frac{UU}{U}\right]\left[\text{One of } \left[\frac{X}{U}\right] \text{ or } \left[\frac{Y}{U}\right]\right] \times 3 \left[\text{One of } \left[\frac{Z}{X}\right] \text{ or } \left[\frac{Z}{Y}\right]\right] \times 3 \left[\frac{E}{ZZZE}\right]$

corresponding to earlier choice

For $\left[\text{One of } \left[\frac{X}{U}\right] \text{ or } \left[\frac{Y}{U}\right]\right] \times 3$, there are $2\times 2\times 2 = 8$ arrangements

$\therefore$ 8 matches with 10 tiles

d. Let p be the number of $\left[\frac{XXXXXXX}{X}\right]$ tiles

Let q be the number of $\left[\frac{X}{XXXXXXXXXX}\right]$ tiles

① $p+q=n$

For a match, the arrangement must be such that:

$$7p+q\times 1 = 1\times p + 10q$$
$$6p = 9q$$
$$p = \frac{3q}{2} \quad ②$$

Sub ② into ①

$$q+\frac{3q}{2} = n$$
$$n = \frac{5q}{2}$$

for p to be an integer, q must be even. If q is even, n is a multiple of 5.

$\therefore$ matches with n tiles exist when n is a multiple of 5.